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QUASIBILINEAR FUNCTIONALS

Abstract. In this paper a certain natural generalization of bilinear functionals is introduced and investigated. We define quasibilinear functionals by replacing the additivity of a bilinear functional with three weaker conditions. The solution is a sequence of bilinear functionals on subspaces of the given linear space.

Quasibilinear functionals are useful in considerations of general projective metrics defined by Rozenfel'd in [3] and generalizations of projective metric spaces as projective spaces with a relation of orthogonality ([2]). These applications will be presented in the next paper.

This work is a part of the author's doctoral thesis entitled "Weak structures of orthogonality on projective spaces of finite dimension" (in Polish), completed under the supervision of doc. dr Edward Siwek at the Silesian University in Katowice. The author wishes to thank dr hab. Marek Kordos for valuable suggestions.

1. Basic notions. In this paper the symbol V always denotes a linear space of a finite dimension n over a commutative field F of characteristic not equal to 2. The zero vector of V is denoted by Θ. We write $ID(u, v, \ldots, w)$ iff vectors $u, v, \ldots, w$ are linearly independent and $D(u, v, \ldots, w)$ otherwise. If U is a linear subspace of V then we write $U < V$. In particular $\emptyset < V$, $\{\Theta\} < V$, $\dim \emptyset = -1$ and $\dim \{\Theta\} = 0$. If $U < V$, $U \neq V$ and $\dim U \geqslant 1$ then we say that U is a proper subspace of V. The linear closure of a set $M \subset V$ is denoted by $\operatorname{Lin} M$. In the case of $M = \{u\}$, $u \neq \Theta$, we write shortly (u) instead of $\operatorname{Lin} \{u\}$. The symbol $\mathrm{d}V$ denotes the set of all directions (i.e. 1-dimensional subspace) of V.

Recall that a *bilinear (symmetric) functional* on V is a mapping $f\colon V \times V \to F$ satisfying the following axioms:

B1 $\forall\ u, v \in V \quad (f(u, v) = f(v, u))$,

B2 $\forall\ u, v \in V \quad \forall\ \lambda \in F \quad (f(u, \lambda v) = \lambda f(u, v))$,

B3 $\forall\ u, v, w \in V \quad (f(u, v+w) = f(u, v) + f(u, w))$.

By $\mathscr{L}_2(V, F)$ we denote the set of all bilinear functionals on V. For arbitrary $f \in \mathscr{L}_2(V, F)$ the structure (V, F) is called an *orthogonal linear space.*

Manuscript received August 10, 1989, and in final form February 26, 1991.

AMS (1980), subject classification: Primary 51N15.

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DEFINITION 1.1. A mapping $f: V\times V\to F$ is called a *quasibilinear functional* on V if, and only if, the following axioms are satisfied:

QB1 $\forall\ u,v\in V \quad (f(u,v)=f(v,u))$,

QB2 $\forall\ u,v\in V \quad \forall\ \lambda\in F \quad (f(u,\lambda v)=\lambda f(u,v))$,

QB3 $\forall\ u,v,w\in V \quad (f(u,v)\neq 0 \wedge f(u,w)\neq 0 \Rightarrow f(u,v+w)=f(u,v)+f(u,w))$,

QB4 $\forall\ u,v,w\in V \quad (f(u,v)=f(u,w)=0 \wedge f(u,v+w)\neq 0 \Rightarrow f(v,v+w)=0)$,

QB5 $\forall\ u,v,w\in V \quad (f(u,v)=f(u,w)=f(v,w)=0 \wedge f(v,v)\neq 0$
$\Rightarrow f(u,v+w)=0)$.

The set of all quasibilinear functionals on V is denoted by $\mathscr{QL}_2(V,F)$. For arbitrary $f\in\mathscr{QL}_2(V,F)$ the structure (V,f) is called a *quasiorthogonal linear space*.

Note that we consider only symmetric functionals. Moreover we have

COROLLARY 1.2. $\mathscr{L}_2(V,F)\subset\mathscr{QL}_2(V,F)$.

Conversely, by an easy verification we obtain

COROLLARY 1.3. *A quasibilinear functional $f\in\mathscr{QL}_2(V,F)$ is bilinear on V if, and only if,*

(B) $\qquad \forall\ u,v,w\in V \quad (f(u,v)=f(u,w)=0\Rightarrow f(u,v+w)=0)$.

Since the axioms QB1—QB5 are universal sentences then we get

COROLLARY 1.4. *If $f\in\mathscr{QL}_2(V,F)$ and $U<V$ then $f|U\times U\in\mathscr{QL}_2(U,F)$.*

DEFINITION 1.5. A quasibilinear functional $f\in\mathscr{QL}_2(V,F)$ is said to be *nondegenerated* if, and only if,

QB6 $\forall\ u\in V\setminus\{\Theta\}\ \exists\ v\in V \quad (f(u,v)\neq 0)$.

The set of all nondegenerated quasibilinear functionals on V is denoted by $\mathscr{NQL}_2(V,F)$. A quasiorthogonal linear space (V,f) is said to be nondegenerated if, and only if, $f\in\mathscr{NQL}_2(V,F)$. Analogically we put $\mathscr{NL}_2(V,F):=\mathscr{L}_2(V,F)\cap\mathscr{NQL}_2(V,F)$.

The conditions QB1—QB6 are a certain version of the conditional Cauchy equation (see e.g. [1]). Moreover, one can easily verify that the axioms QB1—QB6 are independent.

We say that vectors $u,v\in V$ are *orthogonal* with respect to $f\in\mathscr{QL}_2(V,F)$ and we write $u\perp_f v$ iff $f(u,v)=0$. If f is fixed then we write shortly $\perp$ instead of $\perp_f$. The axioms QB1 and QB2 imply the following

COROLLARY 1.6. *If $f\in\mathscr{QL}_2(V,F)$ then*

(i) $\forall\ u,v\in V \quad (u\perp v\Leftrightarrow v\perp u)$,

(ii) $\forall\ u\in V \quad (u\perp\Theta)$,

(iii) $\forall\ u,v\in V \quad (u\perp v\Rightarrow\forall\ \lambda,\mu\in F\ \ (\lambda u\perp\mu v))$.

Let $f\in\mathscr{QL}_2(V,F)$. A nonzero vector $v\in V$ is said to be *isotropic* iff $v\perp v$. The last corollary makes it possible to define an isotropic direction as $A\in dV$ such that A contains an isotropic vector. Subspaces $U,W<V$ are said to be *orthogonal* (we write $U\perp W$) iff $u\perp w$ for every $u\in U$ and $w\in W$. Analogically, we define an orthogonality $u\perp W$ *for* $u\in V$ and $W<V$. Now, by Corollary 1.6,

we can say that a direction $A \in \mathrm{d}V$ is isotropic iff $A \perp A$. Moreover, in the usual manner we define the singularity of subspaces, i.e. a proper subspace $U < V$ is said to be *singular* in (V,f) iff $U \perp V$. We may easily verify the following

COROLLARY 1.7. *A quasibilinear functional f is nondegenerated on V if and only if (V,f) contains no singular subspaces.*

Corollaries 1.6 and 1.7 give some interpretations of the axioms QB1, QB2 and QB6, respectively. Now we shall interpret the remaining axioms. The axiom QB3 implies

COROLLARY 1.8. *If $f \in \mathscr{Q}\mathscr{L}_2(V,F)$, $u, v, w \in V$, $ID(v,w)$, $u \not\perp v$ and $u \not\perp w$ then there exists the unique $A \in \mathrm{dLin}(v,w)$ such that $u \perp A$.*

Proof. Since $u \not\perp v$ and $u \not\perp w$ then, by virtue of QB1, QB2 and QB3, for arbitrary $\lambda, \mu \in F\backslash\{0\}$ we have $f(u, \lambda v+\mu w) = \lambda f(u,v)+\mu f(u,w)$. Hence $f(u, \lambda v+\mu w) = 0$ iff $\lambda = \rho f(u,w)$ and $\mu = -\rho f(u,v)$ for some $\rho \in F$. Thus $A = \big(f(u,w)v - f(u,v)w\big)$.

By QB1—QB4 we obtain

COROLLARY 1.9. *If $f \in \mathscr{Q}\mathscr{L}_2(V,F)$, $u, v, w \in V$, $ID(v,w)$, $u \perp v$, $u \perp w$ and $u \not\perp v+w$ then $u \perp A \perp (v+w)$ for every $A \in \mathrm{dLin}(v,w)\backslash\{(v+w)\}$.*

Proof. It follows from the assumptions and Corollary 1.8 that for every direction $A \in \mathrm{dLin}(v,w)$ different from $(v+w)$ we have $u \perp A$. For an arbitrary vector $t \in \mathrm{Lin}(v,w)\backslash(v+w)$ there exist $\lambda, \mu \in F$ such that $\lambda \neq \mu$ and $t = \lambda v+\mu w$. If $\lambda = 0$ or $\mu = 0$ then by Corollary 1.6 we have $t \perp v+w$. Let $\lambda \neq 0$ and $\mu \neq 0$. Putting $s = (\lambda-\mu)w$ we obtain $t+s \in (v+w)$, but since $u \perp t$, $u \perp s$ and $u \not\perp t+s$ then by QB4 we have $t \perp t+s$. Thus $t \perp v+w$ for every $t \in \mathrm{Lin}(v,w)\backslash(v+w)$.

Finally, the evident interpretation of QB5 gives the following

COROLLARY 1.10. *If $f \in \mathscr{Q}\mathscr{L}_2(V,F)$, $u, v, w \in V$, $ID(v,w)$, $u \perp v \perp w \perp u$ and $v \not\perp v$ then $u \perp \mathrm{Lin}(v,w)$.*

From the point of view of applications even more important than Corollary 1.9 is the following

COROLLARY 1.11. *If $f \in \mathscr{Q}\mathscr{L}_2(V,F)$, $u, v, w \in V$, $ID(v,w)$, $u \in \mathrm{Lin}(v,w)$, $u \perp v$ and $u \perp w$ then $u \perp t$ for every $t \in \mathrm{Lin}(v,w)\backslash(u)$.*

Proof. It follows from Corollary 1.8 that either $u \perp \mathrm{Lin}(v,w)$ or there is a unique direction $A \in \mathrm{dLin}(v,w)$ such that $u \not\perp A$ and $u \perp t$ for every $t \in \mathrm{Lin}(v,w)\backslash A$. Obviously, it is sufficient to consider only the second case. Let $s = \lambda v+\mu w$ and $u \not\perp s$, where $\lambda, \mu \in F$. Hence $\lambda \neq 0$ and $\mu \neq 0$. By Corollary 1.10 we obtain $t \perp s$ for every $t \in \mathrm{Lin}(v,w)\backslash(s)$. Since $s \not\perp u$ then $u \in (s)$, i.e. $(u) = (s) = A$. Thus $u \perp t$ for every $t \in \mathrm{Lin}(v,w)\backslash(u)$.

Given $f \in \mathscr{Q}\mathscr{L}_2(V,F)$ and $U < V$, $\dim U = 2$. Putting $f_1 = f|U \times U$, by Corollary 1.4, we obtain $f_1 \in \mathscr{Q}\mathscr{L}_2(U,F)$. It follows from Corollary 1.3 that if $f_1 \in \mathscr{L}_2(U,F)$, $u, v, w \in U$, $u \neq \Theta$, $ID(v,w)$, $u \perp v$ and $u \perp w$ then $(u) \perp U$, i.e. (u) is a singular subspace of (U,f_1). Now, let us assume $f_1 \in \mathscr{Q}\mathscr{L}_2(U,F)\backslash\mathscr{L}_2(U,F)$. By virtue of Corollary 1.3 there exist $u, v, w \in U$ such that $u \perp v$, $u \perp w$ and $u \not\perp v+w$. Consequently by Corollaries 1.6 and 1.11 we have $ID(v,w)$, $u \neq \Theta$, $(u) = (v+w)$ and $(u) \not\perp (u)$. Thus (u) is not a singular subspace of (U,f_1) but is has a very similar property: $(u) \perp t$ for $t \in U\backslash(u)$. This suggests the following

DEFINITION 1.12. Let $f \in \mathcal{Q}\mathcal{L}_2(V, F)$. A proper subspace $U < V, U \neq V$, is called a *quasisingular subspace* of (V, f) if and only if $U \perp t$ for every $t \in V \setminus U$.

It is evident that each singular subspace of a quasiorthogonal space (V, f) is quasisingular. Before investigating the properties of quasisingular subspaces we must introduce some additional notions.

2. Two-dimensional quasiorthogonal linear spaces. By virtue of Definition 1.1 and Corollary 1.3 we obtain the following

LEMMA 2.1. *If* $\dim V = 1$ *and* $f \in \mathcal{Q}\mathcal{L}_2(V, F)$ *then* $f \in \mathcal{L}_2(V, F)$.

Since for arbitrary vector $e_1 \neq \Theta$ we have $v = v^1 e_1$ for $v \in (e_1)$, where $v^1 \in F$, then this lemma implies

COROLLARY 2.2. *If* $\dim V = 1, f \in \mathcal{Q}\mathcal{L}_2(V, F)$ *and* $e_1 \in V \setminus \{\Theta\}$ *then either*

$$f(u, v) = 0 \quad \text{for } u, v \in \mathrm{V}, \tag{2.2.1}$$

or

$$f(e_1, e_1) \neq 0 \text{ and } f(u, v) = u^1 v^1 f(e_1, e_1) \quad \text{for } u, v \in V. \tag{2.2.2}$$

This implies that if $\dim V = 1$ and $f \in \mathcal{Q}\mathcal{L}_2(V, F)$ then either $u \perp v$ for every $u, v \in V$, or $\forall\, u, v \in V \big(u \perp v \Rightarrow (u = \Theta \vee v = \Theta)\big)$. In the first case the functional f is degenerated, in the second case f is nondegenerated.

It is well known that if $\dim V = 2$ and $f \in \mathcal{L}_2(V, F)$ then there exists a basis $\langle e_1, e_2 \rangle$ of V such that

$$f(u, v) = u^1 v^1 f(e_1, e_1) + u^2 v^2 f(e_2, e_2) \tag{2.2.3}$$
$$\text{for } u = u^1 e_1 + u^2 e_2 \text{ and } v = v^1 e_1 + v^2 e_2 \in V.$$

DEFINITION 2.3. Let $\langle e_1, e_2 \rangle$ be a basis of a 2-dimensional space V and let f be a bilinear functional on V determined by formula (2.2.3). The quasiorthogonal space (V, f) is said to be:

(i) *totally degenerated space* (TDS) iff $f(e_1, e_1) = f(e_2, e_2) = 0$,

(ii) *parabolic space* (PS) iff $f(e_1, e_1) \neq 0$ and $f(e_2, e_2) = 0$, or $f(e_1, e_1) = 0$ and $f(e_2, e_2) \neq 0$,

(iii) *elliptic space* (ES) iff $f(e_1, e_1) \neq 0, f(e_2, e_2) \neq 0$ and

$$\forall\, \lambda, \mu \in F \quad (\lambda^2 f(e_1, e_1) + \mu^2 f(e_2, e_2) = 0 \Rightarrow \lambda = \mu = 0), \tag{2.2.4}$$

(iv) *hyperbolic space* (HS) iff $f(e_1, e_1) \neq 0, f(e_2, e_2) \neq 0$ and

$$\exists\, \lambda, \mu \in F \setminus \{0\} \quad (\lambda^2 f(e_1, e_1) + \mu^2 f(e_2, e_2) = 0). \tag{2.2.5}$$

LEMMA 2.4. *If* $\dim V = 2$ *and* $f \in \mathcal{Q}\mathcal{L}_2(V, F) \setminus \mathcal{L}_2(V, F)$ *then there exists a basis* $\langle e_1, e_2 \rangle$ *of* V *such that*

$$f(e_1, e_1) \neq 0 \text{ and } f(u, v) = \begin{cases} u^1 v^1 f(e_1, e_1) & \text{if } u^2 = v^2 = 0 \\ u^2 v^2 f(e_1, e_2) & \text{if } u^2 \neq 0 \text{ or } v^2 \neq 0. \end{cases} \tag{2.2.6}$$

Proof. From Corollary 1.3 we may see that there exist $e_1, e_2, e_3 \in V$ such that $e_1 \perp e_2, e_1 \perp e_3$ and $e_1 \not\perp e_2 + e_3$. It follows from Corollaries 1.6 and 1.11 that $ID(e_2, e_3)$ and

(1) $$e_1 \perp t \quad \text{for } t \in V \setminus (e_1).$$

Hence $D(e_1, e_2 + e_3)$,

(2) $$e_1 \not\perp e_1,$$

and $ID(e_1, e_2)$, i.e. the vectors e_1, e_2 form the basis of V and for every $u, v \in V$ we have $u = u^1 e_1 + u^2 e_2$ and $v = v^1 e_1 + v^2 e_2$. It results from QB1 and QB2 that

(3) $$\big(u^2 = v^2 = 0 \Rightarrow f(u, v) = u^1 v^1 f(e_1, e_1)\big) \quad \text{for } u, v \in V.$$

Moreover, from (2) we have

(4) $$f(e_1, e_1) \neq 0.$$

Also (1) implies

(5) $$\big((u^2 = 0 \wedge v^2 \neq 0 \vee u^2 \neq 0 \wedge v^2 = 0) \Rightarrow f(u, v) = u^2 v^2 f(e_2, e_2)\big).$$

Now, let $u^2 \neq 0$ and $v^2 \neq 0$. From (5) we have $f(u, v^2 e_2) = 0$, and since $u \perp e_1, v \perp e_1, v^2 e_2 \perp e_1$ and $e_1 \not\perp e_1$, then by Corollary 1.11 we obtain the equivalence

$$f(u, v) = 0 \Leftrightarrow f(u, v^2 e_2) = 0,$$

and consequently, from Corollary 1.3, we have $f(u, v) = f(u, v^2 e_2)$. Analogically we derive $f(u, v^2 e_2) = f(u^2 e_2, v^2 e_2)$, therefore $f(u, v) = f(u^2 e_2, v^2 e_2)$. Thus

(6) $$\big(u^2 \neq 0 \wedge v^2 \neq 0 \Rightarrow f(u, v) = u^2 v^2 f(e_2, e_2)\big) \quad \text{for } u, v \in V,$$

because $f(u^2 e_2, v^2 e_2) = u^2 v^2 f(e_2, e_2)$. This completes the proof.

DEFINITION 2.5. Let $\dim V = 2$, $f \in \mathcal{QL}_2(V, F) \setminus \mathcal{L}_2(V, F)$ and let $\langle e_1, e_2 \rangle$ be a basis of V such that f is determined by the formula (2.2.6). The quasiorthogonal space (V, f) is said to be:
(i) *quasitotally degenerated* (QTDS) iff $f(e_2, e_2) = 0$,
(ii) *quasiparabolic space* (QPS) iff $f(e_2, e_2) \neq 0$.

Now, we have

COROLLARY 2.6. *Let* $\dim V = 2$ *and let* $f \in \mathcal{QL}_2(V, F)$. *A quasiorthogonal space* (V, f) *is either* $TDS, PS, ES, HS, QTDS$, *or* QPS.

This implies that using the notions of isotropic, quasisingular and singular direction and the notions of orthogonal and degenerated spaces we may complete the properties of 2-dimensional quastiorthogonal linear spaces as in the following Table.

TABLE 2.7.

	orthogonal spaces	degenerated spaces	isotropic directions	quasisingular directions	singular directions
TDS	+	+	all	all	all
PS	+	+	exactly one	exactly one	exactly one
ES	+	−	none	none	none
HS	+	−	exactly two	none	none
QTDS	−	+	all except one	all	all except one
QPS	−	−	none	exactly one	none

Moreover, since $\operatorname{char} F \neq 2$ then we have the following

LEMMA 2.8. *If* $\dim V = 2$ *then* V *contains at least four different directions.*

This makes it possible to recognize the type of an arbitrary 2-dimensional quasiorthogonal space (V, f) by the properties of a relation $\perp_f$. For example, if (V, f) contains exactly two isotropic directions then (V, f) is an HS. Analogically we obtain:

COROLLARY 2.9. *If* $\dim V = 2, f \in \mathcal{Q}\mathcal{L}_2(V, F)$, $u, v \in V$, $ID(u, v)$ *and* $u \not\perp v$ *then there exists exactly one direction* $A \in \mathrm{d}V$ *such that* $u \perp A$.

Note that 2-dimensional quasiorthogonal spaces which are not orthogonal may be defined as the spaces containing some quasisingular direction which is not singular. This suggests the following:

DEFINITION 2.10. Let $\dim V = 2$ and $f \in \mathcal{Q}\mathcal{L}_2(V, F)$. A quasisingular direction $A \in \mathrm{d}V$ is called an *axis* of (V, f) iff A is not singular.

Thus

COROLLARY 2.11. *If* $\dim V = 2, f \in \mathcal{Q}\mathcal{L}_2(V, F)$ *and* (V, f) *contains an axis then* (V, f) *is either QTDS or QPS.*

Now, we consider a case of a 2-dimensional subspace U of a quasiorthogonal space (V, f). It follows from Corollary 1.4 that $f|U \times U \in \mathcal{Q}\mathcal{L}_2(U, F)$. By virtue of Corollary 1.10, Table 2.7 and Definition 2.10 we obtain

COROLLARY 2.12. *If* $f \in \mathcal{Q}\mathcal{L}_2(V, F)$, $u, v, w \in V$, $ID(v, w)$, $u \perp v$, $u \perp w$, $U = \mathrm{Lin}(v, w)$ *and*

(i) $(U, f|U \times U)$ *is an ES or an HS*

or

(ii) $(U, f|U \times U)$ *is a PS and* $w \perp w$

or

(iii) (w) *is an axis of* $(U, f|U \times U)$

then $u \perp U$.

LEMMA 2.13. *If* $f \in \mathcal{Q}\mathcal{L}_2(V, F)$, $u, v, w \in V$, $u \perp v$, $u \perp w$ *and* $u \not\perp v + w$ *then there exists a subspace* $U < V$ *and a direction* $A \in \mathrm{d}U$ *such that* $\dim U = 2$, $U < \mathrm{Lin}(u, v, w)$ *and* A *is an axis of* $(U, f|U \times U)$.

Proof. Suppose that this is not true, i.e. there is no 2-dimensional subspace of $\mathrm{Lin}(u, v, w)$ which contains an axis. Hence, by Corollaries 2.6, 2.11 and Table 2.7, for every $U < \mathrm{Lin}(u, v, w)$, $\dim U = 2$, the space $(U, f|U \times U)$ is orthogonal. Then from the hypotheses, Corollary 1.9 and Table 2.7 the vectors u, v, w are linearly independent and the direction $(v+w)$ is singular in $(\mathrm{Lin}(v, w), f|\mathrm{Lin}(v, w) \times \mathrm{Lin}(v, w))$. Therefore $(\mathrm{Lin}(u, v+w), f|\mathrm{Lin}(u, v+w) \times \mathrm{Lin}(u, v+w))$ is an HS because $u \not\perp v+w$ and $v+w \perp v+w$. Hence there exist vectors $t, s \in \mathrm{Lin}(u, v+w) \backslash (v+w)$ such that $t \perp t$ and $s \not\perp s$. Moreover, from Corollary 2.12, for every $p \in \mathrm{Lin}(v, w) \backslash (v+w)$ we have $p \perp \mathrm{Lin}(u, v+w)$. It follows from Table 2.7 that $(\mathrm{Lin}(v, w), f|\mathrm{Lin}(v, w) \times \mathrm{Lin}(v, w))$ is either a TDS or a PS. If this space is a TDS then for any $p \in \mathrm{Lin}(v, w) \backslash (v+w)$ the space $(\mathrm{Lin}(t, p), f|\mathrm{Lin}(t, p) \times \mathrm{Lin}(t, p))$ is a TDS, and from Table 2.7 we obtain $z \perp z$ for every $z \in \mathrm{Lin}(u, v, w) \backslash \mathrm{Lin}(u, v+w)$. Hence $(\mathrm{Lin}(v, s), f|\mathrm{Lin}(v, s) \times \mathrm{Lin}(v, s))$ is a QTDS, because of $s \not\perp s$. This contradicts the hypothesis. Now, let us assume that $(\mathrm{Lin}(v, w), f|\mathrm{Lin}(v, w) \times \mathrm{Lin}(v, w))$ is a PS. Then for any $p \in \mathrm{Lin}(v, w) \backslash (v+w)$ the space $(\mathrm{Lin}(t, p) \times \mathrm{Lin}(t, p))$ is also parabolic, and from Table 2.7 we obtain $z \not\perp z$ for $z \in \mathrm{Lin}(u, v, w) \backslash \mathrm{Lin}(u, v+w)$. Analogically, for any $z \in \mathrm{Lin}(u, v, w) \backslash \mathrm{Lin}(u, v+w)$ the space $(\mathrm{Lin}(v+w, z), f|\mathrm{Lin}(v+w, z) \times \mathrm{Lin}(v+w, z))$ is also parabolic. Therefore, from Corollary 2.12, we have $z \perp \mathrm{Lin}(u, v+w)$ and $z \not\perp z$ for $z \in \mathrm{Lin}(u, v, w) \backslash \mathrm{Lin}(u, v+w)$. Thus $(\mathrm{Lin}(s, v), f|\mathrm{Lin}(s, v) \times \mathrm{Lin}(s, v))$ is a QPS which again contradicts the hypothesis. This completes the proof.

This lemma suggests the following

DEFINITION 2.14. Let $f \in \mathscr{2L}_2(V, F)$. A direction $A \in \mathrm{d}V$ is called an *axis* of (V, f) if and only if there exists a subspace $U < V$ such that $\dim U = 2$, $A \in \mathrm{d}U$ and A is an axis of the space $(U, f|U \times U)$.

From Corollary 1.3, Definitions 1.12, 2.10, 2.14 and Lemma 2.13 we deduce the following

COROLLARY 2.15. *If $f \in \mathscr{2L}_2(V, F)$ then the space (V, f) is orthogonal if and only if (V, f) does not contain any axes.*

3. A relation of an axial orthogonality. In the preceding section we proved that the searching of all vectors $u, v, w \in V$ such that $u \perp v$, $u \perp w$ and $u \not\perp v+w$ may be replaced by the searching of all axes. Moreover, it follows from Corollary 1.11 that an axis (s) of 2-dimensional space (V, f) may be defined as such direction $(s) \in \mathrm{d}V$ that $u \perp s$, $s-u \perp s$ and $s \not\perp s$ for some $u \in V$. This suggests the following

DEFINITION 3.1. Let $f \in \mathscr{2L}_2(V, F)$. We say that a vector $u \in V$ is *axially orthogonal* to a vector $s \in V$ and we write $u \llcorner s$ if and only if $u \perp s$, $s-u \perp s$ and $s \not\perp s$.

By virtue of Corollaries 1.7 and 1.11 we obtain the following

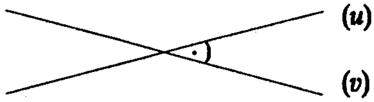
COROLLARY 3.2. *If $f \in \mathscr{2L}_2(V, F)$, $u, s \in V$ and $u \llcorner s$ then $ID(u, s)$* and $s \perp t$ *for every $t \in \mathrm{Lin}(u, s) \backslash (s)$.*

Since $s \not\perp s$ then the last condition means that (s) is an axis of the space $(\mathrm{Lin}(u,s), f|\mathrm{Lin}(u,s)\times\mathrm{Lin}(u,s))$. Moreover, from Table 2.7 and Corollary 2.9 we derive

COROLLARY 3.3. *If* $f\in\mathcal{Q}\mathcal{L}_2(V,F)$, $u,v,s\in V$, $u \llcorner v$, $u \llcorner s$ *and* $ID(v,s)$ *then* $ID(u,v,s)$.

To illustrate the further considerations we shall give figures based on the following:

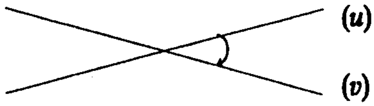
REMARK 3.4. Let V be a linear space over a commutative field F and let $\dim V=3$. A direction $A\in \mathrm{d}V$ may be represented by a line on the projective plane $\mathcal{P}$ over F. Then 2-dimensional subspaces $U<V$ may be treated as points on $\mathcal{P}$. Moreover for an arbitrary 2-dimensional subspace $U<V$ and a vector $u\in V\backslash\{\Theta\}$ "the line" (u) passes through "the point" U iff $u\in U$. The orthogonality of the directions (u) and (v) we denote as follows:

(i) 

if $ID(u,v)$ and $u\perp v$,

(ii) 

if $u\neq\Theta$ and $u\perp u$,

(iii) 

if $ID(u,v)$ and $t\perp v$ for $t\in\mathrm{Lin}(u,v)\backslash(v)$.

Now we shall investigate further properties of the relation $\llcorner$. At the beginning, let us note that according to Corollaries 1.6, 3.2, Table 2.7 and Definition 3.1 the relation $\llcorner$ is neither reflexive nor symmetric. However, it is transitive, i.e. we have

COROLLARY 3.5. *If* $f\in\mathcal{Q}\mathcal{L}_2(V,F)$, $u,v,w\in V$, $u\llcorner v$ *and* $v\llcorner w$ *then* $u\llcorner w$.

Proof. It follows from Table 2.7, Definition 3.1 and Corollary 3.2 that $ID(u,v,w)$, $v\not\perp v$ and $w\not\perp w$. By virtue of Corollary 2.12 we obtain the following alternative

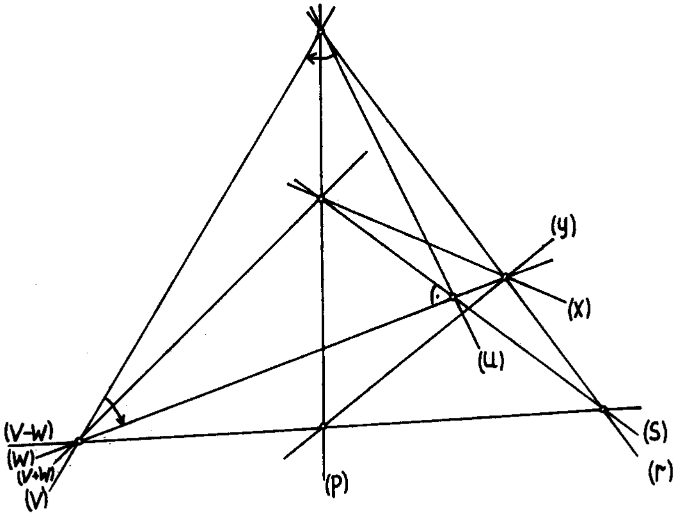
$$(1)\qquad (\forall\ t\in\mathrm{Lin}(u,v)\backslash(v)\quad (w\not\perp t))\vee(w\perp\mathrm{Lin}(u,v)).$$

Suppose that

$$(\mathrm{Hp})\qquad \forall\ t\in\mathrm{Lin}(u,v)\backslash(v)\quad (w\not\perp t).$$

Then $w\not\perp u$ and by Corollary 2.9 there exists exactly one $(s)\in \mathrm{dLin}(u,w)$ such that $w\perp(s)$. Let us put (Fig. 1):

$$p\in\mathrm{Lin}(s,v+w)\cap\mathrm{Lin}(u,v)\backslash\{\Theta\},$$
$$r\in\mathrm{Lin}(s,v-w)\cap\mathrm{Lin}(u,v)\backslash\{\Theta\},$$
$$x\in\mathrm{Lin}(p,v+w)\cap\mathrm{Lin}(w,r)\backslash\{\Theta\},$$
$$y\in\mathrm{Lin}(p,v-w)\cap\mathrm{Lin}(w,r)\backslash\{\Theta\}.$$



Since $v+w \perp w \perp s$ and $w \not\perp p$, then from Corollary 1.9 we have $w \perp x$. Analogically we derive $w \perp y$. Thus $x \llcorner w$ and consequently $r \perp w$, which contradicts (Hp). Therefore (1) implies

(2) $$w \perp \mathrm{Lin}(u, v).$$

It follows from the assumptions, (2) and Corollary 2.12 that

(3) $$u+v \perp \mathrm{Lin}(v, w).$$

Moreover, $(\mathrm{Lin}(v, w), f|\mathrm{Lin}(v, w) \times \mathrm{Lin}(v, w))$ is a QPS because $v \llcorner w$ and $v \not\perp v$. Hence by (3) and Corollary 1.10 we deduce $w \perp \mathrm{Lin}(u+v, v+w)$. Now, for $t \in \mathrm{Lin}(u, w) \cap \mathrm{Lin}(u+v, v+w) \setminus \{\Theta\}$ we have $w \perp t$ and $ID(u, t)$. Therefore $u \llcorner w$, because $u \perp w, t \perp w, w \not\perp w$ and $t \in \mathrm{Lin}(u, w) \setminus (u)$.

In Definition 1.12 we introduced the notion of a quasisingular subspace. It is obvious that an axis of a 2-dimensional space (V, f) is such a subspace and it is not a singular subspace. The next example is given by the following

LEMMA 3.6. *If* $f \in \mathscr{Q}\mathscr{L}_2(V, F), u, v, w \in V, u \llcorner v, u \llcorner w$ *and* $ID(v, w)$ *then* $t \perp \mathrm{Lin}(v, w)$ *for* $t \in \mathrm{Lin}(u, v, w) \setminus \mathrm{Lin}(v, w)$.

Proof. It follows from the assumptions and Corollary 3.3 that

(1) $$ID(u, v, w).$$

Consequently, from Corollary 3.2 we have

(2) $$\forall\ t \in \mathrm{Lin}(u, v) \setminus (v) \quad (t \perp v),$$

(3) $$\forall\ t \in \mathrm{Lin}(u, w) \setminus (w) \quad (t \perp w).$$

Moreover

(4) $$v \not\perp v$$

and

(5) $$w \not\perp w.$$

Since $u \perp w$ then (2), Corollaries 1.10 and 1.11 imply the alternative

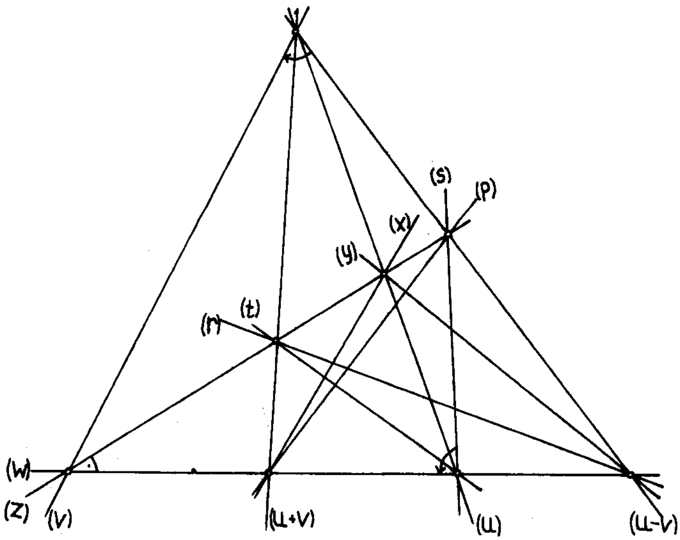
(6) $$(\forall\, t \in \mathrm{Lin}(u, v)\backslash(v) \quad (t \perp w)) \;\vee\; (\forall\, t \in \mathrm{Lin}(u, v)\backslash(u) \quad (t \not\perp w)).$$

Now, suppose that

(Hp) $$\forall\, t \in \mathrm{Lin}(u, v)\backslash(u) \quad (t \not\perp w).$$

In particular $v \not\perp w$ and by Corollary 2.9 there is a unique $(z) \in \mathrm{dLin}(v, w)$ such that $w \perp z$. Moreover (5) implies $ID(w, z)$. Let us put (Fig. 2):

$$t \in \mathrm{Lin}(u, w) \cap \mathrm{Lin}(z, u+v)\backslash\{\Theta\},$$
$$s \in \mathrm{Lin}(u, w) \cap \mathrm{Lin}(z, u-v)\backslash\{\Theta\},$$
$$r \in \mathrm{Lin}(w, u-v) \cap \mathrm{Lin}(z, u+v)\backslash\{\Theta\},$$
$$p \in \mathrm{Lin}(w, u+v) \cap \mathrm{Lin}(z, u-v)\backslash\{\Theta\},$$
$$x \in \mathrm{Lin}(w, u+v) \cap \mathrm{Lin}(z, u)\backslash\{\Theta\},$$
$$y \in \mathrm{Lin}(w, u-v) \cap \mathrm{Lin}(z, u)\backslash\{\Theta\},$$



It follows from (3) and the definition of t that $t \perp w$ and, since $w \perp z$ and $w \not\perp u+v$, then Corollary 1.10 implies $r \perp w$. Analogically we obtain

(H1) $$p \perp w.$$

Moreover, by Corollary 1.10 we derive an alternative

(H2) $$x \perp w \vee y \perp w$$

because $u \perp w$ and $z \perp w$. Now, with the help of (5), (H1), (H2) and 3.1 we obtain the alternative $(x \,\llcorner\, w \vee y \,\llcorner\, w)$ which implies

(H3) $$u+v\perp w \vee u-v\perp w.$$

Since the sentences (Hp) and (H3) are contradictory then from (6) we deduce

(7) $$\forall\, t\in \mathrm{Lin}(u, v)\backslash(v) \quad (t\perp w)$$

and, from the symmetry of the assumptions with respect to v and w, we have

(8) $$\forall\, t\in \mathrm{Lin}(u, w)\backslash(w) \quad (t\perp v).$$

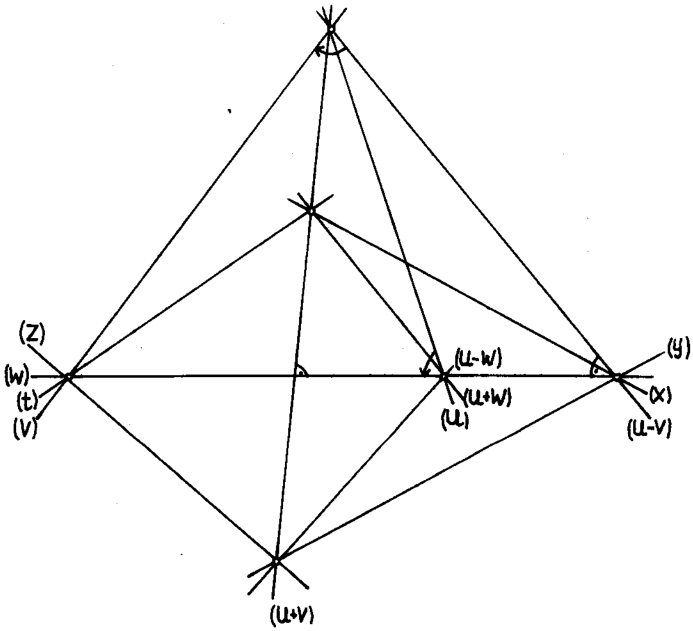
Now, let us put (Fig. 3):

$$t\in \mathrm{Lin}(v, w)\cap \mathrm{Lin}(u+v, u+w)\backslash\{\Theta\},$$

$$z\in \mathrm{Lin}(v, w)\cap \mathrm{Lin}(u+v, u-w)\backslash\{\Theta\},$$

$$x\in \mathrm{Lin}(u-v, w)\cap \mathrm{Lin}(u+v, u+w)\backslash\{\Theta\},$$

$$y\in \mathrm{Lin}(u-v, w)\cap \mathrm{Lin}(u+v, u-w)\backslash\{\Theta\},$$



The conditions (7) and (8) give $u+w\perp w$ and $u+v\perp w$, hence from Corollary 1.10 we obtain the alternative

(9) $$t\perp w \vee x\perp w.$$

Analogically we derive

(10) $$z\perp w \vee y\perp w.$$

The conjunction $t\perp w \wedge z\perp w$ implies $v\perp w$, and from the alternative $x\perp w \vee y\perp w$ follows $u-v \mathrel{\text{↳}} w$. Thus

(11) $$v \mathrel{\text{↳}} w \vee u-v \mathrel{\text{↳}} w$$

and from the symmetry of the assumptions

(12) $$w \llcorner v \vee u-w \llcorner v.$$

Let $s \in \mathrm{Lin}(u, v, w) \setminus \mathrm{Lin}(v, w)$. Then $s = \lambda u+\mu v+\rho w$, where $\lambda, \mu, \rho \in F$ and $\lambda \neq 0$. If $\mu = 0$ or $\rho = 0$ then it follows from (2), (3), (7) and (8) that $s \perp w$ and $s \perp v$. Now, let us assume that $\mu \neq 0$ and $\rho \neq 0$, and let us consider two possible cases:

(i) $v \perp w$. Since (3) and (8) imply $\lambda u+\rho w \perp w$ and $\mu v \perp \lambda u+\rho w$, respectively, then from (4) and Corollary 1.11 we deduce $w \perp \lambda u+\mu v+\rho w$, i.e. $w \perp s$. From the symmetry we also get $v \perp s$.

(ii) $v \not\perp w$. Then (11) and (12) imply $u-v \llcorner w$ and $u-w \llcorner v$, and consequently $\lambda u-\mu v+\rho w \perp w$. Now, since $\lambda u+\rho w \perp w$ and $\mu v \not\perp w$, then from Corollaries 1.10 and 3.2 we have $\lambda u+\rho w \llcorner \mu v$. This implies $s \perp v$. Analogically we obtain $s \perp w$.

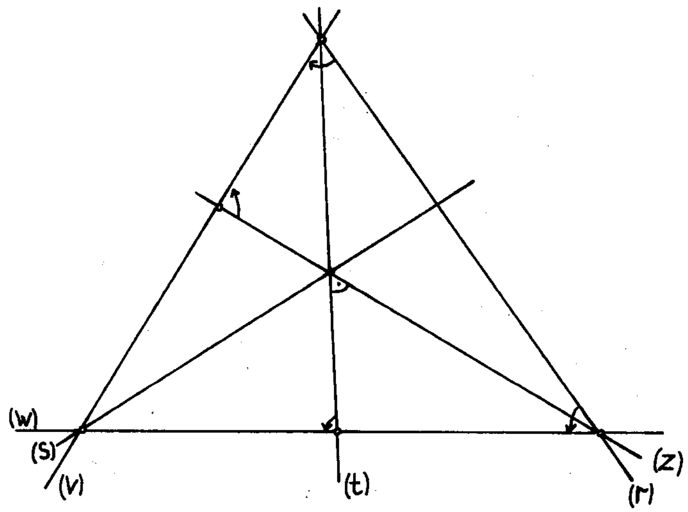
Thus

(13) $$\forall\, t \in \mathrm{Lin}(u, v, w) \setminus \mathrm{Lin}(v, w) \quad (v \perp t \perp w).$$

Now, let us fix arbitrary vectors $t \in \mathrm{Lin}(u, v, w) \setminus \mathrm{Lin}(v, w)$ and $s \in \mathrm{Lin}(v, w)$. Suppose that

(Hp) $$t \not\perp s.$$

Then (13) implies $s \notin (v)$ and $s \notin (w)$, and by virtue of Corollary 2.9 there exists a unique $(z) \in \mathrm{dLin}(t, s)$ such that $t \perp (z)$. Assuming additionally $D(t, z)$, from Table 2.7 we derive that $(\mathrm{Lin}(t, s), f|\mathrm{Lin}(t, s) \times \mathrm{Lin}(t, s))$ is an HS and all the spaces $(\mathrm{Lin}(t, w), f|\mathrm{Lin}(t, w) \times \mathrm{Lin}(t, w))$, $(\mathrm{Lin}(v, t+w), f|\mathrm{Lin}(v, t+w) \times \mathrm{Lin}(v, t+w))$, $(\mathrm{Lin}(v, t-w), f|\mathrm{Lin}(v, t-w) \times \mathrm{Lin}(v, t-w))$ are quasitotally degenerated. Since the directions (t), $\mathrm{Lin}(s, t) \cap \mathrm{Lin}(v, t+w)$, $\mathrm{Lin}(s, t) \cap \mathrm{Lin}(v, t-w)$ are three distinct isotropic directions in $(\mathrm{Lin}(s, t), f|\mathrm{Lin}(s, t) \times \mathrm{Lin}(s, t))$, then from Table 2.7 this space cannot be an HS. This contradiction implies $ID(t, z)$ and consequently $t \not\perp t$. Now, let us put (Fig. 4):



$$r \in \mathrm{Lin}(w, z) \cap \mathrm{Lin}(v, t) \setminus \{\Theta\}.$$

Since $t \perp z \perp w \perp t$ and $w \not\perp w$, then from Corollary 1.10 we have $t \perp r$ and, because $t \llcorner v$, the space $(\mathrm{Lin}(t, v), f|\mathrm{Lin}(t, v) \times \mathrm{Lin}(t, v))$ is a QTDS and $t \perp t$. This contradiction gives

$$\forall\, t \in \mathrm{Lin}(u, v, w) \setminus \mathrm{Lin}(v, w) \quad \forall s \in \mathrm{Lin}(v, w) \quad (t \perp s).$$

With the help of Corollary 3.5 we may generalize the above lemma as follows:

LEMMA 3.7. *If* $f \in \mathcal{QL}_2(V, F), u, v, w, s \in V, ID(u, v, w),\ s \in \mathrm{Lin}(u, v, w)$, $u \llcorner v$ *and* $s \llcorner w$ *then* $t \perp \mathrm{Lin}(v, w)$ *for every* $t \in \mathrm{Lin}(u, v, w) \setminus \mathrm{Lin}(v, w)$.

Moreover,

LEMMA 3.8. *If* $f \in \mathcal{QL}_2(V, F), u, v, w, s \in V, ,u \llcorner v, s \llcorner w$ *and* $ID(u, v, w, s)$ *then* $u \llcorner w$ *or* $s \llcorner v$.

Proof. We consider two possibilities:

(i) $v \perp w$. Then $v+s \perp w$ or $v-s \perp w$ and without restricting the generality we may assume that $v+s \perp w$. From the assumptions we have $w-s \perp w$, hence $(v+s)+(w-s) \perp w$ or $(v+s)-(w-s) \perp w$. If $(v+s)+(w-s) \perp w$ (i.e. $v+w \perp w$) then $v \llcorner w$, which from Corollary 3.5 gives $u \llcorner w$. Not let us assume that $(v+s)+(w-s) \not\perp w$ (i.e. $v+w \not\perp w$). Then $(v+s)-(w-s) \perp w$ and it follows from Corollary 1.9 that $v+w \perp v+s$ and $v+w \perp w-s$. Further we infer that $v+s \perp v$ or $v+w \perp v$, and $w-s \perp v$ or $v+w \perp v$. If $v+w \perp v$ then $w \llcorner v$, hence by virtue of Corollary 2.12 we deduce $s \perp v$, but now $v+s \perp v$ and $v \not\perp v$ imply $s \llcorner v$.

(ii) $v \not\perp w$. Then, by virtue of Corollary 2.9, there exists a unique direction $(z) \in \mathrm{dLin}(v, w)$ such that $w \perp (z)$. Since $s+z \perp w$ or $s-z \perp w$, then without restricting the generality we may assume that $s+z \perp w$. Moreover, $s \llcorner w$ implies $s+w+z \perp w$ or $s+w-z \perp w$ and analogically $s-w+z \perp w$ or $s-w-z \perp w$. If $s+w-z \perp w$ and $s-w-z \perp w$ then $s-z \llcorner w$ and putting $t \in \mathrm{Lin}(w, s-z) \cap \mathrm{Lin}(v, s) \setminus \{\Theta\}$ we obtain $t \perp w$ and further $s \llcorner v$. If $s+w+z \perp w$ or $s-w+z \perp w$ then $s+z \llcorner w$ and putting $t \in \mathrm{Lin}(w, s+z) \cap \mathrm{Lin}(v, s) \setminus \{\Theta\}$ we obtain $t \perp w$, which again implies $s \llcorner v$.

Now, applying the above lemmas, by induction on a number of axes we may prove the three lemmas.

LEMMA 3.9. *If* $f \in \mathcal{QL}_2(V, F)$, $k \in \mathrm{N}$, $u_1, \ldots, u_k, v_1, \ldots, v_k \in V, ID(v_1, \ldots, v_k)$ *and* $u_1 \llcorner v_1, \ldots, u_k \llcorner v_k$ *then*

$$\exists\, m \in \{1, \ldots, k\} \quad \forall i \in \{1, \ldots, k\} \quad (u_m \llcorner v_i).$$

LEMMA 3.10. *If* $f \in \mathcal{QL}_2(V, F)$, $k \in \mathrm{N}, u, v_1, \ldots, v_k \in V, ID(v_1, \ldots, v_k)$ *and* $u \llcorner v_1, \ldots, u \llcorner v_k$ *then* $u \perp \mathrm{Lin}(v_1, \ldots, v_k) \perp u+t$ *for* $t \in \mathrm{Lin}(v_1, \ldots, v_k)$.

LEMMA 3.11. *If* $f \in \mathcal{QL}_2(V, F)$, $k \in \mathrm{N}, u, v_1, \ldots, v_k \in V, ID(v_1, \ldots, v_k)$ *and* $u \llcorner v_1, \ldots, u \llcorner v_k$ *then* $ID(u, v_1, \ldots, v_k)$.

4. The representation theorem. In this section we shall investigate connections between quasisingular supspaces of (V, f) defined in Definition 1.12 and the relation $\llcorner$. At the beginning we have.

DEFINITION 4.1. Let $f \in \mathcal{Q}\mathcal{L}_2(V, F)$. The set of all vectors of axes is the set

$$q(V, f) := \{v \in V: \quad \exists\ u \in V\ (u \llcorner v)\}. \tag{4.1.1}$$

We recall that $\dim V = n$. Now, from Lemma 3.11 we have

COROLLARY 4.2. *If* $f \in \mathcal{Q}\mathcal{L}_2(V, F)$ *and* $q(V, f) \neq \varnothing$ *then* $\dim \mathrm{Lin}(q(V, f)) < n$.

Moreover, Definitions 1.12, 2.10, 2.14, 3.1, 4.1 and Corollaries 2.15, 3.2, 4.2 imply the following important

COROLLARY 4.3. *If* $f \in \mathcal{Q}\mathcal{L}_2(V, F)$ *then the following conditions are equivalent*:

(i) (V, f) *is not orthogonal*,

(ii) $1 \leqslant \dim \mathrm{Lin}(q(V, f)) \leqslant n-1$.

Consequently, with the help of the lemmas given in the preceding section we may easily deduce the following

COROLLARY 4.4. *If* $f \in \mathcal{Q}\mathcal{L}_2(V, F)$ *and* $q(V, f) \neq \varnothing$ *then* $t \perp \mathrm{Lin}(q(V, f))$ *for every* $t \in V \setminus \mathrm{Lin}(q(V, f))$.

The above corollary shows that the subspace $\mathrm{Lin}(q(V, f))$ is quasisingular in (V, f). Moreover, this subspace is not singular because all vectors $v \in q(V, f)$ are not isotropic. More precisely: if $\dim \mathrm{Lin}(q(V, f)) = k$ then there are vectors $v_1, \ldots, v_k \in q(V, f)$ such that $v_1 \not\perp v_1, \ldots, v_k \not\perp v_k$ and $\mathrm{Lin}(v_1, \ldots, v_k) = \mathrm{Lin}(q(V, f))$. This suggests the following

DEFINITION 4.5. Let $f \in \mathcal{Q}\mathcal{L}_2(V, F)$. A subspace $W < V$ is called a *properly quasisingular subspace* of (V, f) if and only if either $W = \varnothing$, or W is a quasisingular subspace of (V, f) generated by nonisotropic vectors. A maximal (in the sense of inclusion) properly quasisingular subspace of (V, f) will be denoted by $S(V, f)$.

As a simple consequence of Definitions 1.12, 4.1, 4.5 and Corollaries 4.2—4.4 we obtain the following important corollary.

COROLLARY 4.6. *If* $f \in \mathcal{Q}\mathcal{L}_2(V, F)$ *then*

$$S(V, f) < V, \tag{4.6.1}$$

$$V \neq \varnothing \Rightarrow S(V, f) \neq V, \tag{4.6.2}$$

$$S(V, f) = \begin{cases} \mathrm{Lin}(q(V, f)) & \textit{if } q(V, f) \neq \varnothing, \\ \varnothing & \textit{if } q(V, f) = \varnothing, \end{cases} \tag{4.6.3}$$

$$\dim V \leqslant 1 \Rightarrow S(V, f) = \varnothing, \tag{4.6.4}$$

$$\forall u \in V \setminus S(V, f) \quad (u \perp S(V, f)) \tag{4.6.5}$$

and

$$\forall u \in V \setminus S(V, f) \quad \forall v \in S(V, f) \quad (v \not\perp v \Rightarrow u \llcorner v). \tag{4.6.6}$$

Moreover, we may easily verify the following

COROLLARY 4.7. *If* $f \in \mathcal{Q}\mathcal{L}_2(V, F), u, v, w \in V$ *then*

$$v, w, v+w \notin S(V, f) \Rightarrow f(u, v+w) = f(u, v) + f(u, w) \tag{4.7.1}$$

and

$$(4.7.2)\qquad u, v \notin S(V, f) \wedge w \in S(V, f) \Rightarrow f(u, v+w) = f(u, v).$$

Since by virtue of Corollary 1.4 and Definition 4.5 for any $f \in \mathcal{Q}\mathcal{L}_2(V, F)$ a pair $(S(V, f), f|S(V, f) \times S(V, f))$ is a quasiorthogonal space as well, then we may consider $S(S(V, f), f|S(V, f) \times S(V, f))$. Thus we may define the following family of subspaces of V:

$$(4.8.1)\quad S_i(V, f) := \begin{cases} V & \text{when } i = 0, \\ S(S_{i-1}(V, f), f|S_{i-1}(V, f) \times S_{i-1}(V, f)) & \text{when } i \in \mathrm{N}. \end{cases}$$

Now, the conditions (4.6.1)—(4.6.4) imply

$$(4.8.2)\qquad V = S_0(V, f) > S_1(V, f) > S_2(V, f) > \dots$$

and

$$(4.8.3)\qquad \forall i \in \mathrm{N} \quad (S_{i-1}(V, f) \neq \emptyset \Rightarrow S_{i-1}(V, f) \neq S_i(V, f)),$$

whence one can deduce (because the dimension of V is finite) that there exists a unique natural number τ such that $S_\tau(V, f) \neq \emptyset$ and $S_{\tau+1}(V, f) = \emptyset$.

DEFINITION 4.8. Let $f \in \mathcal{Q}\mathcal{L}_2(V, F)$. The natural number $\tau(f)$ determined by the condition

$$(4.8.4)\qquad \tau(f) := \min\{i \in \mathrm{N}\colon\ S_i(V, f) = \emptyset\}$$

is called the *type of* the *functional f.*

Note that

$$(4.8.5)\qquad 1 \leqslant \tau(f) \leqslant \dim V$$

and by Corollaries 4.3 and 4.6

$$(4.8.6)\qquad \tau(f) = 1 \Leftrightarrow f \in \mathcal{L}_2(V, F).$$

Now, for the simplicity, for a given functional $f \in \mathcal{Q}\mathcal{L}_2(V, F)$ we put

$$(4.8.7)\qquad \tau := \tau(f),$$

$$(4.8.8)\qquad V_i := S_{\tau-i}(V, f) \quad \text{for } i = 0, 1, \dots, \tau,$$

$$(4.8.9)\qquad f_i := f|V_i \times V_i \quad \text{for } i = 1, \dots, \tau.$$

According to (4.8.2), (4.8.4), (4.8.7) and (4.8.8) we have

$$(4.8.10)\qquad \emptyset = V_0 < V_1 < \dots < V_{\tau-1} < V_\tau = V.$$

Note that (4.8.6) and (4.8.9) imply

$$(4.8.11)\qquad f_1 \in \mathcal{L}_2(V_1, F).$$

DEFINITION 4.9. Let $f \in \mathcal{Q}\mathcal{L}_2(V, F)$ and let $\tau(f) = \tau$. The mappings $f^1\colon V_1 \times V_1 \to F, \dots, f^\tau\colon V_\tau \times V_\tau \to F$ uniquely determined by

$$(4.9.1) \qquad f^i(u,v) := \begin{cases} f_i(u,v) & \text{when } u \notin V_{i-1} \quad \text{or} \quad v \notin V_{i-1}, \\ 0 & \text{when } u, v \in V_{i-1}, \end{cases}$$

for $i = 1, \dots, \tau$, are called *components of* the *functional f*.

Now, using Corollaries 4.6, 4.7, relations (4.8.1)—(4.8.4), (4.8.7), (4.8.8) and Definition 4.9 we easily obtain the following

COROLLARY 4.10. *The components of a quasibilinear functional are bilinear.*

Moreover, from Definition 4.5, Corollary 4.6, relations (4.8.7)—(4.8.9) and Definition 4.9 we derive.

COROLLARY 4.11. *If* $f \in \mathcal{Q}\mathcal{L}_2(V,F)$, $\tau = \tau(f)$, $\tau > 1$ *and* $i \in \{2, \dots, \tau\}$ *then* V_{i-1} *is the singular subspace of the orthogonal space* (V_i, f^i).

Finally we can formulate the following representation theorem.

THEOREM 4.12. *Let V be an n-dimensional vector space over a commutative field F of characteristic different from 2. A mapping* $f\colon V \times V \to F$ *is a quasibilinear functional on V if and only if there exist a natural number* τ, *subspaces* $V_0, V_1, \dots, V_\tau$ *of V and symmetric bilinear functionals* $f^1\colon V_1 \times V_1 \to F, \dots, f^\tau\colon V_\tau \times V_\tau \to F$ *such that the following conditions are satisfied*:

(i) $$1 \leqslant \tau \leqslant \dim V,$$

(ii) $$\varnothing = V_0 < V_1 < \dots < V_{\tau-1} < V_\tau = V,$$

(iii) $$\dim V \geqslant 1 \Rightarrow \dim V_1 \geqslant 1,$$

(iv) $$\forall i \in \{2, \dots, \tau\} \quad (V_{i-1} \neq V_i),$$

(v) $$\forall i \in \{2, \dots, \tau\} \quad \forall u \in V_{i-1} \quad \forall v \in V_i \quad (f^i(u,v) = 0),$$

(vi) $$\forall i \in \{1, \dots, \tau - 1\} \quad \exists\, u, v \in V_i \quad (f^i(u,v) \neq 0)$$

and

(vii) *if* $u, v \in V_i$, $u \notin V_{i-1}$ *or* $v \notin V_{i-1}$, *then* $f(u,v) = f^i(u,v)$ *for* $i = 1, \dots, \tau$ *and* $u, v \in V$.

Proof. The implication "$\Rightarrow$" results from Definitions 4.5, 4.9, Corollaries 4.10, 4.11, relations (4.8.3), (4.8.7), (4.8.8) and (4.8.10). On the other hand, the implication "$\Leftarrow$" can be obtained by an easy verification of Definition 1.1.

Moreover, by an easy verification, one can obtain the following

COROLLARY 4.13. *Let* $f \in \mathcal{Q}\mathcal{L}_2(V,F)$, $n = \dim V$, $\tau = \tau(f)$ *and let* $f^1, \dots, f^\tau$ *be components of f. If* $r_1, \dots, r_\tau$ *are the ranges of the bilinear functionals* $f^1, \dots, f^\tau$ *then*

$$f \in \mathcal{N}\mathcal{Q}\mathcal{L}_2(V,F) \Leftrightarrow r_1 + \dots + r_\tau = n.$$

Appendix 1. Congruent quasibilinear functionals. Since for any bilinear functional there exists an orthogonal basis of V then we can formulate the representation theorem in the analytical manner as follows:

COROLLARY. *A mapping $f\colon V\times V\to F$ is a quasibilinar functional on V if and only if there exist a natural number $\tau(f)=\tau$ called the type of f, a basis $\{e_i\}_{i=1,\dots,n}$ of V, natural numbers $n_1,\dots,n_\tau$ and scalars $\varepsilon_1,\dots,\varepsilon_n\in F$ such that*

(i) $$1\leqslant \tau\leqslant n,$$

(ii) $$0=:n_0<n_1<\dots<n_\tau=n,$$

(iii) $$\forall k\in\{1,\dots,\tau-1\}\quad \exists\, i\in\{n_{k-1}+1,\dots,n_k\}\quad (\varepsilon_i\neq 0)$$

and

(iv) *if $u_i=v_i=0$ for $i=n_k+1,\dots,n$ and there exists $i\in\{n_{k-1}+1,\dots,n_k\}$ such that $u_i\neq 0$ or $v_i\neq 0$ then $f(u,v)=\varepsilon_{n_{k-1}+1}u_{n_{k-1}+1}v_{n_{k-1}+1}+\dots+\varepsilon_{n_k}u_{n_k}v_{n_k}$ for $k=1,\dots,\tau$ and $u=(u_1,\dots,u_n)$, $v=(v_1,\dots,v_n)\in V$.*

Moreover, f is nondegenerated if and only if $\varepsilon_i\neq 0$ for $i=1,\dots,n$.

Each quasibilinear functional f in an orthogonal (with respect to f) basis $\{e_i\}_{i=1,\dots,n}$ is uniquely determined by the sequence $((\varepsilon_1,\dots,\varepsilon_{n_1}),\dots,(\varepsilon_{n_{\tau-1}+1},\dots,\varepsilon_n))$ and we denote shortly this canonical form of the functional f by $(f)_{\langle e_1,\dots,e_n\rangle}$, i.e.

$$(f)_{\langle e_1,\dots,e_n\rangle}=((\varepsilon_1,\dots,\varepsilon_{n_1}),\dots,(\varepsilon_{n_{\tau-1}+1},\dots,\varepsilon_n)).$$

Now we adopt the following

DEFINITION. We say that two quasibilinear functionals f,g of V are *congruent* and we write $f\simeq g$ if and only if f and g determine the same relation of orthogonality of vectors, i.e.

$$f\simeq g\Leftrightarrow \perp_f=\perp_g\Leftrightarrow \forall u,v\in V\,(f(u,v)=0\Leftrightarrow g(u,v)=0).$$

In other words congruent functionals determine the same structure $(V,\perp)$, where $\perp=\perp_f=\perp_g$, called a *weakly quasiorthogonal linear space.* Since a functional f is bilinear if and only if the conjunction $u\perp_f v$ and $u\perp_f w$ implies $u\perp_f v+w$ for every $u,v,w\in V$, then for any congruent quasibilinear functionals f,g on V the bilinearity of f is equivalent to the bilinearity of g. Moreover, it is evident that if quasibilinear functionals f,g on V are congruent and if $\{e_i\}_{i=1,\dots,n}$ is some basis of V orthogonal with respect to f, then this basis is orthogonal with respect to g as well.

Let f,g be congruent quasibilinear functionals on V and let $\{e_i\}_{i=1,\dots,n}$ be a basis of V orthogonal with respect to f. By virtue of Corollary there are natural numbers $\tau=\tau(f)$, $\alpha=\tau(g)$, $n_1,\dots,n_\tau$, $m_1,\dots,m_\alpha$ and scalars $\varepsilon_1,\dots,\varepsilon_n$, $\omega_1,\dots,\omega_n$ such that $(f)_{\langle e_1,\dots,e_n\rangle}=((\varepsilon_1,\dots,\varepsilon_{n_1}),\dots,(\varepsilon_{n_{\tau-1}+1},\dots,\varepsilon_n))$ and $(g)_{\langle e_1,\dots,e_n\rangle}=((\omega_1,\dots,\omega_{m_1}),\dots,(\omega_{m_{\alpha-1}+1},\dots,\omega_n))$. Now putting $f_k:=f|(\mathrm{Lin}(e_{n_{k-1}+1},\dots,e_n))^2$ and $g_k:=g|(\mathrm{Lin}(e_{n_{k-1}+1},\dots,e_n))^2$ for $k=1,\dots,\tau$ we see that functionals $f_k|(\mathrm{Lin}(e_{n_{k-1}+1},\dots,e_{n_k}))^2$, $g_k|(\mathrm{Lin}(e_{n_{k-1}+1},\dots,e_{n_k}))^2$ are bilinear and the functionals $f_k|(\mathrm{Lin}(e_{n_{k-1}+1},\dots,e_{n_k},e_{n_k+1}))^2$, $g_k|(\mathrm{Lin}(e_{n_{k-1}+1},\dots,e_{n_k},e_{n_k+1}))^2$ are not bilinear for $k=1,\dots,\tau-1$ from which we deduce in turn $n_1\leqslant m_1,\dots,n_{\tau-1}\leqslant m_{\tau-1}$, $n_\tau\leqslant m_\tau$ and analogically $m_1\leqslant n_1,\dots,m_{\alpha-1}\leqslant n_{\alpha-1}$, $m_\alpha\leqslant n_\alpha$ and consequently $\tau=\alpha$ and $n_k=m_k$ for $k=1,\dots,\tau$. Thus we have $(f)_{\langle e_1,\dots,e_n\rangle}=((\varepsilon_1,\dots,\varepsilon_{n_1}),\dots,(\varepsilon_{n_{\tau-1}},\dots,\varepsilon_n))$ and $(g)_{\langle e_1,\dots,e_n\rangle}=((\omega_1,\dots,\omega_{n_1}),\dots,(\omega_{n_{\tau-1}+1},\dots,\omega_n))$. Let us note that the restrictions $f^1:=f|(\mathrm{Lin}(e_1,\dots,e_{n_1}))^2$,

$g^1 := g|(\mathrm{Lin}(e_1,\dots,e_{n_1}))^2$ are bilinear and $f^1 \simeq g^1$. Moreover $(f^1)_{\langle e_1,\dots,e_{n_1}\rangle} = ((\varepsilon_1,\dots,\varepsilon_{n_1}))$ and $(g^1)_{\langle e_1,\dots,e_{n_1}\rangle} = ((\omega_1,\dots,\omega_{n_1}))$. If $\varepsilon_1 = \dots = \varepsilon_{n_1} = 0$ then of course $\omega_1 = \dots = \omega_{n_1} = 0$ and $\omega_1 = \lambda_1\varepsilon_1,\dots,\omega_{n_1} = \lambda_1\varepsilon_{n_1}$ for any $\lambda_1 \in F\backslash\{0\}$. Now, let us assume additionally that $\varepsilon_j \neq 0$ for some $j\in\{1,\dots,n_1\}$. Therefore $\omega_j \neq 0$ as well and we may put $\lambda_1 := \omega_j(\varepsilon_j)^{-1}$. We consider two possibilities. If $n_1 = 1$ then of course $g^1 = \lambda_1 f^1$. In the second case, if $n_1 > 1$, then for every $i\in\{1,\dots,n_1\}\backslash\{j\}$ the inequality $\varepsilon_i \neq 0$ implies $\omega_i \neq 0$, $f^1(e_i - e_j, \varepsilon_j e_i + \varepsilon_i e_j) = 0$ and $g^1(e_i - e_j, \varepsilon_j e_i + \varepsilon_i e_j) = 0$ because $f^1 \simeq g^1$. Since $g^1(e_i - e_j, \varepsilon_j e_i + \varepsilon_i e_j) = \omega_i\varepsilon_j - \omega_j\varepsilon_i = \varepsilon_j(\omega_i - \lambda_1\varepsilon_i) = 0$ then $\omega_i = \lambda_1\varepsilon_i$. Thus for every $i\in\{1,\dots,n_1\}$ we have $\omega_i = \lambda_1\varepsilon_i$ and consequently $g^1 = \lambda_1 f^1$.

Analogically, putting $(f^2)_{\langle e_1,\dots,e_{n_2}\rangle} = ((0,\dots,0,\varepsilon_{n_1+1},\dots,\varepsilon_{n_2}))$ and $(g^2)_{\langle e_1,\dots,e_{n_2}\rangle} = ((0,\dots,\omega_{n_1+1},\dots,\omega_{n_2}))$ we deduce that $f^2 \simeq g^2$ and that there is $\lambda_2\in F\backslash\{0\}$ such that $\omega_{n_1+1} = \lambda_2\varepsilon_{n_1+1},\dots,\omega_{n_2} = \lambda_2\varepsilon_{n_2}$. Continuing this procedure we obtain the following:

THEOREM. *If f, g are quasibilinear functionals on V, $\{e_i\}_{i=1,\dots,n}$ is a basis of V orthogonal with respect to f and if $(f)_{\langle e_1,\dots,e_n\rangle} = ((\varepsilon_1,\dots,\varepsilon_{n_1}),\dots,(\varepsilon_{n_{\tau-1}+1},\dots,\varepsilon_n))$ then f, g are congruent if, and only if, the basis $\{e_i\}_{i=1,\dots,n}$ is orthogonal with respect to g and there exist scalars $\lambda_1,\dots,\lambda_\tau\in F\backslash\{0\}$ such that $(g)_{\langle e_1,\dots,e_n\rangle} = ((\lambda_1\varepsilon_1,\dots,\lambda_1\varepsilon_{n_1}),\dots,(\lambda_\tau\varepsilon_{n_{\tau-1}+1},\dots,\lambda_\tau\varepsilon_n))$.*

Appendix 2. Independence of axioms.

THEOREM. *The axioms* QB1,...,QB6 *are independent.*

Proof. To prove our theorem, for each QBi, where $i\in\{1,\dots,6\}$, we give a suitable field F_i, a vector space V_i and a mapping $f_i\colon V_i\times V_i \to F_i$ such that f_i satisfies all the axioms QB1,...,QB6 except QBi. Since for every $i\in\{1,\dots,6\}$ it is easy to verify that (V_i, f_i) is a model of the axiom system $\{$QB1,...,QB6$\}\backslash\{$QB$i\}$, then we show only that f_i does not satisfy QBi.

Firstly, for QB1 we adopt $F_1 = \mathbf{R}$, $V_1 = \mathbf{R}\times\mathbf{R}$ and $f_1(u,v) = u^1v^2 - u^2v^1$ for $u = (u^1,u^2)$, $v = (v^1,v^2)\in V_1$. Then putting $u = (1,0)$ and $v = (0,1)$ we have $f_1(u,v) = 1$ and $f_1(v,u) = -1$. Thus QB1 does not hold.

For QB2 we put $F_2 = \mathbf{Q}(\sqrt{2})$, $V_2 = F_2\times F_2$ and $f_2(u,v) = \overline{u^1}\,\overline{v^1} + \overline{u^2}\,\overline{v^2}$ for $u,v\in V_2$, where $\overline{a+b\sqrt{2}} = a - b\sqrt{2}$ for $a,b\in\mathbf{Q}$. Now, putting $u = v = (1,0)$ and $\lambda = 1+\sqrt{2}$ we obtain $f_2(u,\lambda v) = 1-\sqrt{2}$ and $\lambda f_2(u,v) = 1+\sqrt{2}$ in spite of QB2.

In all the remaining cases we adopt $F = \mathbf{R}$ and $V = \mathbf{R}\times\mathbf{R}$, i.e. $F_i = \mathbf{R}$, $V_i = \mathbf{R}\times\mathbf{R}$ for $i = 3,4,5,6$. Moreover, we put $f_3(u,v) = \sqrt{(u^1)^2+(u^2)^2}\cdot\sqrt{(v^1)^2+(v^2)^2}\cdot\mathrm{sgn}(u^1v^1+u^2v^2)$ for $u,v\in V_3$ and $u = v = (1,0)$, $w = (1,1)$ and we see that $f_3(u,v) = 1 \neq 0$, $f_3(u,w) = \sqrt{2} \neq 0$ and $f_3(u,v+w) = \sqrt{5} \neq f_3(u,v) + f_3(u,w)$. Consequently, if we adopt

$$f_4(u,v) = \begin{cases} 0 & \text{when } u^2v^2 \neq 0 \\ u^2 & \text{when } v^2 = 0 \\ v^2 & \text{when } u^2 = 0 \end{cases}$$

for $u, v \in V_4$, $u=(0,1)$, $v=(0,1)$, $w=(1,-1)$ we obtain $f_4(u,v)=f_4(u,w)=0$, $f_4(u, v+w)=1\neq 0$ and $f_4(v, v+w)=1\neq 0$. Further, putting

$$f_5(u,v)=\begin{cases} u^1v^1+u^2v^2 & \text{when } u, v \text{ are linearly dependent} \\ 0 & \text{when } u, v \text{ are linearly independent} \end{cases}$$

for $u, v \in V_5$, $u=(1,0)$, $v=(1,-1)$, $w=(0,1)$ and we see that $f_5(u,v)=f_5(u,w)=f_5(v,w)=0, f_5(v,v)=2\neq 0$ and $f_5(u,v+w)=1\neq 0$. Finally, a mapping f_6 such that $f_6(u,v)=0$ for $u, v \in V_6$ is a degenerated quasibilinear functional. This completes the proof of our theorem.

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